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Bifurcation of conical magnetic field

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A strong magnetic field develops via bifurcation in a flow with accretion, as analytical and numerical solutions of magnetohydrodynamic equations show. Examples are swirl-free and swirling jets, and thermal convection near a point source of heat and gravity. The features of the resulting conical similarity magnetic flows are analogous to those observed in cosmic jets.

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I. INTRODUCTION

Appearance of a magnetic field in a magnetic-free flow of an electrically conducting fluid is an important manifestation of symmetry breaking of both fundamental and practical interest. Here we report how magnetic field emerges via bifurcation in conically similar solutions of magnetohydrodynamic (MHD) equations.

Conical solutions describe a wide family of jet-like flows where velocity **v** and magnetic field **H** have the representation, $\{v_r, v_\theta, v_\phi\} = \nu r^{-1}\{-\psi', -\psi/\sin\theta, \Gamma/\sin\theta\}$, $\{H_r, H_\theta, H_\phi\} = h\nu r^{-1}\{\Theta', \Theta/\sin\theta, \Phi/\sin\theta\}$; $\{r, \theta, \phi\}$ are spherical coordinates, ν is viscosity, h is a scale constant; ψ, Γ, Θ, and Φ are functions of $x = \cos\theta$ only, and the prime denotes differentiation.

Since **v** and **H** are unbounded at $r=0$, conical solutions describe practical jets only at some distances from the flow source, as Fig. 1 illustrates. Similarity is absent in regions 1 (near nozzle 4) and 3 (near ambient body 5) but occurs in region 2, $r_i < r < r_o$ ($r_o/r_i \gg 1$ for strong jets).

This paper addresses jet-like flows in their similarity region (2 in Fig. 1) and shows that **H** appears via bifurcation as the strength of a magnetic-free flow exceeds a threshold. In the primary flow, **H** develops due to instability and then saturates to a steady magnitude. In the case where both the secondary MHD and primary magnetic-free flows are conical, the analysis of bifurcation (involving full MHD equations) is easier than the analysis of kinematic and nonlinear instabilities. Therefore, it is convenient to first explore the **H** bifurcation and features of the resulting MHD flows—the subject of this paper—and to postpone the instability study for further research. Some comments concerning the instability are in Sec. VI.

II. PROBLEM FORMULATION

Conical similarity is a feature of swirl-free (Landau-Squire) and swirling (Long) jets, thermal convection near a point source of heat and gravity [1], and many other magnetic-free and MHD flows [2].

In the problem on convection near a point source of gravity and heat (Fig. 2), $\mathbf{g} = -\mathbf{e_r}\delta/r^2$, the representation for temperature is $T = T_\infty + \gamma r^{-1}\vartheta(x)$; δ and γ characterize the gravity (**g**) and heat flux (***q***) magnitudes, and $\mathbf{e}_r$ indicates the outward radial direction. In addition, the Boussinesq approximation, $\rho/\rho_\infty = 1 - \beta(T - T_\infty)$ is used; ρ is the fluid density and β is the coefficient of thermal expansion. Conical similarity reduces the MHD equations to

$$(1-x^2)\psi' + 2x\psi^{-1/2}\psi^2 = F - \Theta^2/2, \tag{1a}$$

$$(1-x^2)F''' = \mathrm{Ra}\ \psi\vartheta + 2\Gamma\Gamma' - 2\Phi\Phi', \tag{1b}$$

$$(1-x^2)\Gamma'' = \psi\Gamma' - \Theta\Phi', \tag{1c}$$

$$(1-x^2)\vartheta' = \mathrm{Pr}\ \psi\vartheta, \tag{1d}$$

$$(1-x^2)\Theta'' = \mathrm{Bt}(\psi\Theta' - \psi'\Theta), \tag{1e}$$

$$(1-x^2)\Phi'' = \mathrm{Bt}[\psi\Phi' - \Theta\Gamma' + 2(\psi'\Phi - \Theta'\Gamma) + 2x(\psi\Phi - \Theta\Gamma)/(1-x^2)], \tag{1f}$$

where $\mathrm{Ra} = \beta\gamma\delta/(\kappa\nu)$, $\mathrm{Pr} = \nu/\kappa$, and $\mathrm{Bt} = \nu/\nu_m$ are the Rayleigh, Prandtl, and magnetic Prandtl numbers; κ and ν_m are the thermal and magnetic diffusivities. This reduction to or-

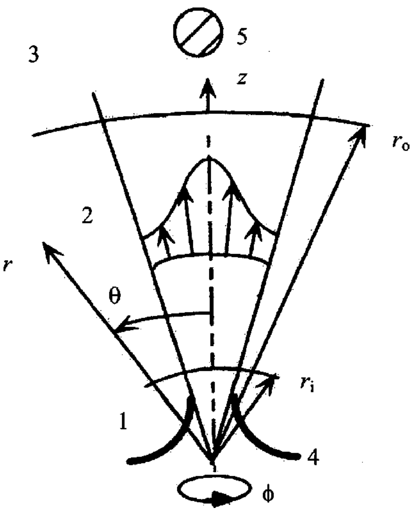

FIG. 1. Schematic of the similarity region in a jet.

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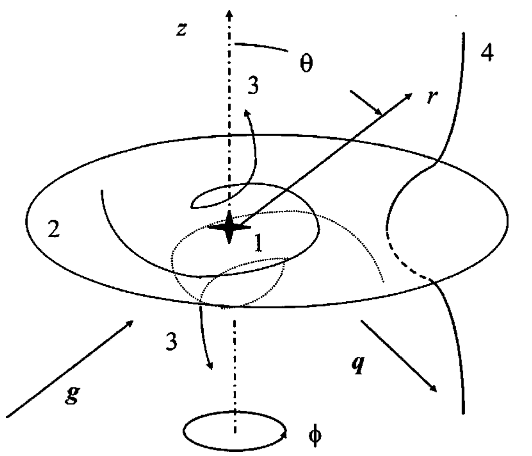

FIG. 2. Problem schematic: a point source (1) of gravity $\boldsymbol{g}$ and heat flux $\boldsymbol{q}$, accretion disk (2), typical stream (3), and magnetic (4) lines.

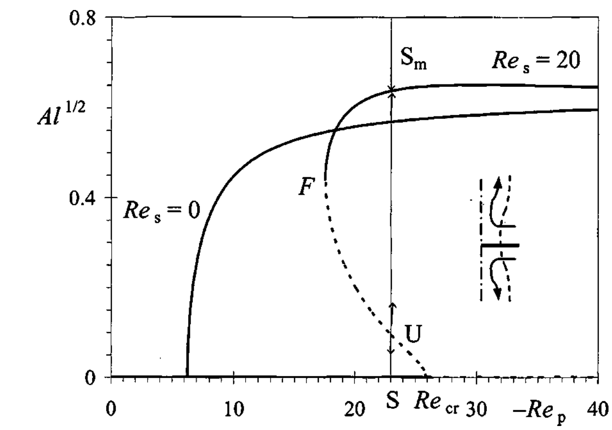

FIG. 3. Bifurcation of magnetic field in accretion ($\mathrm{Re}_s=0$) and vortex-accretion ($\mathrm{Re}_s=20$) flows. The abscissa is a line of the plot symmetry. $\mathrm{Bt}=0.4$.

dinary differential equations radically eases the analysis of MHD bifurcations described below.

III. NECESSARY CONDITION FOR BIFURCATION TO OCCUR

First, we show that magnetic bifurcation occurs in any bipolar flow converging near the disk (2 in Fig. 2). For a prescribed swirl-free flow, where $\Gamma\equiv 0$ and ψ is known, the linear bifurcation problem reduces to $(1e)$ alone. The regularity of $\mathbf{v}$ and $\mathbf{H}$ on the z axis implies $\psi(1)=F'(1)=\Theta(1)=0$ and the symmetry conditions are $\Theta'(0)=\psi(0)=0$. In addition, we use the normalization, $\Theta'(1)=-1$.

Integrating $(1e)$ from $x=0$ to $x=1$ yields that

$$\Theta'(0) = -\int_0^1 2\Theta dx - 2Bt\int_0^1 \psi\Theta' dx. \qquad (2)$$

It is evident now that the bifurcation cannot occur in a diverging flow (where $\psi<0$) because the condition $\Theta'(0)=0$ cannot be satisfied for $\Theta>0$ and $\Theta'<0$ in $0<x<1$ (the problem is invariant with respect to the transformation, $\Theta\to-\Theta$).

In contrast, the bifurcation must occur in a converging flow (where $\psi>0$, Fig. 3). Indeed, $\Theta=1-x$ and $\Theta'(0)=-1$ at $\mathrm{Bt}=0$, while as $\mathrm{Bt}\to\infty$, Θ' becomes proportional to ψ' so that $\Theta'(0)\to\psi'(0)/\psi'(1)$. Since $\psi(0)=\psi(1)=0$ and $\psi>0$ in $0<x<1$, $\psi'(0)$ and $\psi'(1)$ have opposite signs so that $\Theta'(0)>0$ for $\mathrm{Bt}\gg 1$. Being a continuous function of Bt, $\Theta'(0)$ must change its sign as Bt increases from 0 to ∞.

Thus bifurcation of $\mathbf{H}$ cannot occur in a diverging flow and must occur in a converging flow. A few examples follow.

IV. DEVELOPMENT OF MAGNETIC JET IN A VORTEX-ACCRETION FLOW

Consider a vortex-sink motion of the disk material, i.e., the boundary conditions, $\psi=0$, $\psi'=-\mathrm{Re}_p$, and $\Gamma=\mathrm{Re}_s$ at $x=0$. The Reynolds numbers, Re_p and Re_s, characterize the strength of the meridional motion and swirl, respectively. The flow converges to the axis near the disk and goes to infinity along the axis as the inset in Fig. 3 depicts (for a swirl-free flow). As $-\mathrm{Re}_p$ increases, a strong bipolar jet develops near the axis, while the flow near the disk remains comparatively low speed.

The MHD problem reduces to Eq. (1) with (1–d) omitted and $\mathrm{Ra}=0$. The solution of the nonlinear MHD problem reveals that the magnetic field appears via a pitchfork supercritical bifurcation, as the $\mathrm{Re_s}=0$ curve in Fig. 3 shows where the Alfven number, Al, is the magnetic-to-kinetic energy ratio on the disk. Solutions with $\mathrm{Al}^{1/2}>0$ and $\mathrm{Al}^{1/2}<0$ have just opposite directions of the magnetic field. The inset in Fig. 3 illustrates that the generation of the magnetic field occurs mostly near the disk where streamlines (solid) intersect magnetic lines (dashed). Near the axis (i.e., inside the jet), stream and magnetic lines are almost parallel.

For strong accretion ($-\mathrm{Re_p}\to\infty$), the asymptotic solution, $\mathrm{Al}=\mathrm{Bt}$, $\psi=[1-x-\exp(-\mathrm{BtRe_p}x)]/\mathrm{Bt}$, and $\Theta=1-x-[1-\exp(-\mathrm{BtRe_p}x)]/\mathrm{Bt}$, show that the jet is suppressed and the kinetic energy of accretion transforms mainly into magnetic energy.

The addition of swirl on the disk causes important features: (i) hysteretic transitions between magnetic and magnetic-free states, (ii) a local maximum of circulation near the axis, and (iii) the development of both near-axis and near-plane jets as the vortex accretion intensifies.

The $\mathrm{Re_s}=20$ curve in Fig. 3 illustrates feature (i). The bifurcation is now subcritical and there are three solutions between the bifurcation $\mathrm{Re_{cr}}$ and fold (F) points: stable magnetic-free flow S, unstable U, and stable S_m MHD flows (arrows indicate the evolution direction of disturbed states). Transitions between states S and S_m as Re_p varies are hysteretic. In state S, increasing swirl reverses the flow near the axis.

An important feature is that the magnetic field can avert the swirl-induced flow reversal near axis. Another important feature is that the magnetic field suppresses the swirl in the bulk flow and generates the swirl near the axis [via the last term in (1–c)].

As the vortex accretion intensifies at $\mathrm{Re}_s=\mathrm{Re}_p$, a strong swirling magnetic jet develops. Figure 4 illustrates this fea-

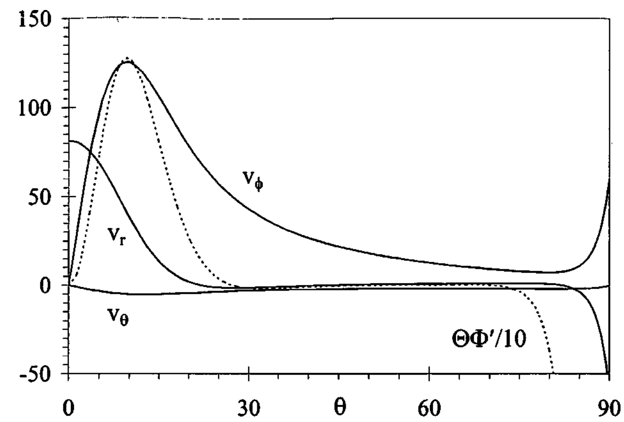

FIG. 4. The velocity (all components are scaled by ν/r) and the magnetic-source-of-swirl ($\Theta\Phi'$) distributions at $\mathrm{Bt}=0.4$ and $-\mathrm{Re}_p=\mathrm{Re}_s=60$.

ture (iii). The swirl velocity near the axis is more than twice v_ϕ prescribed on the disk. The Lorenz force drives this strong swirl as the $\Theta\Phi'$ curve shows. Thus, the self-induced magnetic field suppresses swirl near the disk and generates swirl near the axis. Despite the strong swirl, no flow reversal occurs and the jet is kept collimated by the magnetic field.

V. MAGNETIC BIFURCATION NEAR A POINT SOURCE OF HEAT AND GRAVITY

While the accretion is prescribed in Sec. IV, now it is driven by the buoyancy force (characterized by Ra).

For swirl-free states and $\mathrm{Pr}=0$, system (1) reduces to (1–a), (1–e), and (1–b) with $\vartheta\equiv 1$ and $\Gamma=\Phi\equiv 0$. The boundary conditions are $\psi=F'=\Theta'=0$ (symmetry) at $x=0$ and $\psi=F=F'=\Theta=0$ (regularity) at $x=1$. Note that $\psi'(1)(=-\mathrm{Re}_a)$ cannot be calculated from (1–a) (due to the 0/0 indeterminacy) and must be found to satisfy the boundary conditions.

Figure 5 shows the development of a bipolar MHD convection (Fig. 2). For $\mathrm{Ra}<24$, the fluid is at rest state where the pressure gradient balances the buoyancy force. At $\mathrm{Ra}=24$, bifurcation of magnetic-free thermal convection occurs (b_c in Fig. 5) [1]. As $-\mathrm{Re}_p$, increases along curve $b_c b_m$, a high-speed near-axis jet develops and the dimensionless ve-

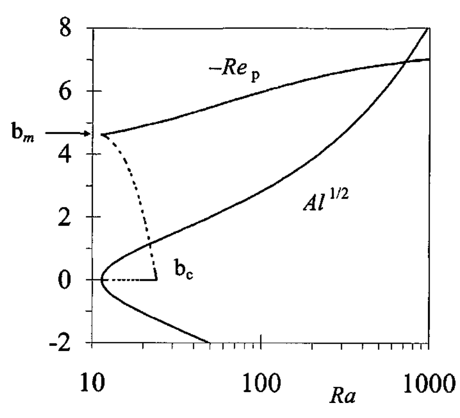

FIG. 5. Bifurcation of thermal convection (b_c) and magnetic field (b_m) near a point source of heat and gravity.

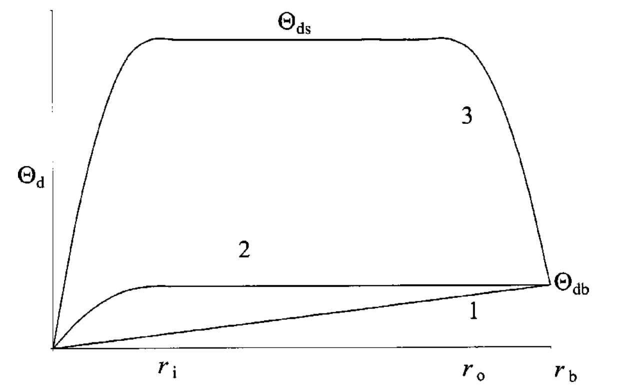

FIG. 6. Sketch of magnetic field development as the jet strength increases.

locity on the axis $\mathrm{Re}_a\to\infty$ while the near-disk flow remains low speed ($\mathrm{Re}_p\to -4.617$). Substituting the corresponding asymptotic solution

$$\psi=x[(4+\mathrm{Re}_p)x^2-\mathrm{Re}_p]\mathrm{Re}_a(1-x)/[4+\mathrm{Re}_a(1-x)]$$

in (1–e) and solving the eigenvalue problem, we find the critical value of $\mathrm{Bt}=0.253$, at which a magnetic field emerges in the high-speed bipolar outflow.

The numerical solution of the nonlinear MHD problem at $\mathrm{Bt}=0.253$ (solid curves in Fig. 5) reveals a pitchfork bifurcation (b_m) of magnetic field (curve $\mathrm{Al}^{1/2}$). In the MHD state, Re_a is bounded at finite Ra and the flow structure is similar to that shown by the inset in Fig. 3.

As Ra increases, the induced magnetic field (Θ) grows while stream function (ψ) remains bounded. We deduce the asymptotic equations, as $\mathrm{Ra}\to\infty$, by neglecting the left-hand-side terms in (1–a), i.e., taking $\Theta^2=2F$, and rescaling: $R=U\,\mathrm{Ra}/\mathrm{Bt}$ and $\psi=y/\mathrm{Bt}$. Then (1b) and (le) transform into $(1-x^2)U'''=y$ and $y'=1/2yU'/U+1/4(1-x^2)[(U'/U)^2-2U''/U]$.

The boundary conditions are $U'=y=0$ at $x=0$ and $U=U'=0$ at $x=1$. This problem has a solution (in addition to $U=y\equiv 0$) with $U(0)=0.1042$ and $U=A(1-x)^2-B(1-x)^{2+n}+$ high-order terms near $x=1$ where $A=0.591$, $B=0.676$, and $n=0.685$.

According to this asymptotic solution, Re_a and Θ both grow proportionally to $\mathrm{Ra}^{1/2}$ as $\mathrm{Ra}\to\infty$. Thus, even a strong magnetic field does not suppress the buoyancy jet (in contrast to the accretion jet in Sec. IV).

VI. CONCLUDING REMARKS

Analytical and numerical solutions of the MHD equations show that bifurcation of magnetic field is typical of conical flows with accretion. It is proved that accretion is a necessary condition for the magnetic bifurcation to occur (Sec. III).

When a converging motion of the disk material drives the flow, the self-induced magnetic field eventually (as accretion intensifies) suppresses the near-axis jet (Sec. IV). In contrast, the jet velocity and magnetic field both increase with the buoyancy force in thermal convection flows (Sec. V). Swirl makes transitions hysteretic between the magnetic-free and

MHD states. The self-induced magnetic field focuses swirl near the disk and jet and keeps the jet collimated even at a strong swirl (Sec. IV).

These features resemble those observed in cosmic jets [3,4]. Though the flows studied here are very different from cosmic jets, the effects of accretion, swirl, and magnetic field on the development of strong jets seem generic.

The appearance of magnetic field found here differs from that in the classical dynamo theory (e.g., Ref. 5). A reason is that $\mathbf{H} \sim 1/r$ here while $\mathbf{H}$ decays significantly faster as $r \to \infty$ in dynamo. Moreover, the bifurcations found here manifest the appearance of *conical* **H**, while some background **H** may exist even before the bifurcation occurs, as Fig. 6 illustrates.

Suppose that magnetic field is prescribed and uniform at $r \geqslant r_b$. Figure 6 sketches the radial distribution of Θ_d which is the $|r\mathbf{H}/(h\nu)|$ value on the disk for subcritical (curve 1), critical (2), and supercritical (3) values of Ra. Accordingly, Θ_d is less, equal, and larger than its prescribed background value, Θ_{db}. In case 3, the saturated value, Θ_{ds} inside the similarity region, $r_{\mathrm{i}} < r < r_{\mathrm{o}}$, is independent of and can significantly exceed Θ_{db}.

The development shown in Fig. 6 agrees with the results of the spatial instability study for steady disturbances [6] and is analogous to the swirl development via bifurcation in electrosprays [7]. Analysis of unsteady perturbations is a subject for further research.

Therefore, even being different from dynamo, the magnetic bifurcation reported here is an important effect explaining an abrupt and dramatic growth of **H** in the similarity region due to increasing inflow of magnetic energy, as the flow strength exceeds its critical value.

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